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The cooling of low-heat-resistance cylinders by radiation

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SUMMARY

The method of matched asymptotic expansions is applied to the non-linear radiative cooling of finite or semi-infinite cylinders. It is shown that the method applies when radiation is the limiting factor in the heat-transfer process, i.e. when the heat resistance of the bulk is relatively low. The analysis will be of importance in the fields of crystal growth and the cooling of fins.

1. Introduction

In recent years there has been a growing interest in the radiative cooling of finite or semi-infinite cylinders, especially in the field of crystal growth. A common way to grow large mono-crystals is to pull the crystal slowly from a melt. The two techniques most widely used are the Czochralski method and the floating-zone method. The second method is mainly used for the growth of silicon crystals. A description of both can be found in a monograph by Brice [1].

The temperatures at which these processes take place are usually rather high. Silicon, for example, a material which is of foremost importance in semiconductor technology, solidifies at 1693 K. This is why the crystal will lose most of its heat by radiation. Heat-transfer problems in this field are therefore invariably non-linear. Another source of non-linearity is the thermal conductivity k. In the case of silicon k is inversely proportional to the absolute temperature over a wide range of temperatures. The remaining material properties are usually independent of the temperature.

For a semi-infinite silicon crystal with a curved solid-liquid interface the problem can therefore be defined as follows. The flow of heat is governed by the differential equation

$$\frac{1}{\bar{r}}\frac{\partial}{\partial \bar{r}}\left(\bar{r}k_m\frac{\bar{T}_m}{\bar{T}}\frac{\partial \bar{T}}{\partial \bar{r}}\right)+\frac{\partial}{\partial \bar{x}}\left(k_m\frac{\bar{T}_m}{\bar{T}}\frac{\partial \bar{T}}{\partial \bar{x}}\right)-v\rho c_p\frac{\partial \bar{T}}{\partial \bar{x}}=0, \tag{1.1}$$

where the temperature field is assumed to be stationary. Here the subscript m refers to conditions at the melting point, $\bar{T}$ is the temperature, $\bar{r}$ denotes the distance from the axis of symmetry, $\bar{x}$ is the axial coordinate, ρ is the density, c_p is the specific heat and v is the axial velocity of the crystal. The ranges of $\bar{x}$ and $\bar{r}$ are restricted by $\bar{x}_0(\bar{r})\leqslant \bar{x}<\infty$ and $0\leqslant \bar{r}\leqslant R$, where $\bar{x}_0(\bar{r})$ denotes the solid-liquid interface, which we shall consider as given in the present problem definition. The boundary conditions are

$$\bar{T} = \bar{T}_m \quad \text{at} \quad \bar{x} = \bar{x}_0(\bar{r}), \tag{1.2}$$

$$\bar{T} \to \bar{T}_0 \quad \text{if} \quad \bar{x} \to \infty, \tag{1.3}$$

$$\frac{\partial \bar{T}}{\partial \bar{r}} = 0 \quad \text{at} \quad \bar{r} = 0, \tag{1.4}$$

$$-\frac{k_m \bar{T}_m}{\bar{T}} \frac{\partial \bar{T}}{\partial \bar{r}} = \sigma \sigma_r (\bar{T}^4 - \bar{T}_0{}^4) \text{ at } \bar{r} = R, \tag{1.5}$$

where σ is the Stefan-Boltzmann constant, σ_r is the emissivity and $\bar{T}_0$ the temperature of the surroundings.

The boundary condition (1.5) seems to be most appropriate when the floating-zone technique is applied. In that case the surface of the crystal is only facing the cooled inner wall of the tank that encloses the experimental area. When the Czochralski method is being used, there will be an influx of heat from the walls of the crucible and the surface of the melt. Corrections can be made in the manner described by Arizumi and Kobayashi [2]. It can be shown that heat losses due to convection are negligible in comparison with radiation losses at temperatures close to the melting point. On the other hand, if $\bar{T}$ approaches $\bar{T}_0$, a correction for convection has to be made. However, it would seem that most of the heat-transfer process will take place in the high-temperature range. Since we are interested in this range only [3], a correction for convection would be an unnecessary complication. For silicon it can be shown that radiation heat transfer inside the crystal is small in comparison with conduction heat transfer. We have therefore neglected this effect.

Methods for solving this problem can be classified in two groups. Those belonging to the first group aim at finding analytical solutions, mostly by means of series expansions [4, 5]. To this end an attempt is made to linearize the radiation condition (1.5) with respect to the temperature $\bar{T}_0$. It is argued that under certain conditions one can take for $\bar{T}_0$ some average temperature of the surrounding gas, and this may be considerably higher than room temperature. Furthermore the analyses are usually applied to crystals that solidify at relatively low temperature, such as germanium ($\bar{T}_m = 1210$ K). Very often, however, these conditions are not met. Silicon crystals that are pulled by the floating-zone technique are usually surrounded by argon, so that there is no radiative exchange between the gas and the crystal. The temperatures too are much higher, so that linearization is out of the question. In the second class of methods finite differences are used [2,6]. Although these methods are rather time-consuming, there is no objection to the inclusion of non-linearities. It is therefore possible to arrive at a solution that more closely approximates to reality. A difference between the analytical and the numerical papers is that the latter seldom give many details of the method employed. They concentrate on the presentation of graphical and tabulated results, but it is not always obvious to a prospective user how he should proceed to get results that apply to his particular situation. The analytical papers, on the other hand sometimes lead to concise and useful results that apply to restricted parts of the temperature field [4].

It will therefore be desirable to have analytical solutions for the non-linear cases. The way to set about this is to introduce dimensionless variables:

$$T = \bar{T}/\bar{T}_m, \; r = \bar{r}/R, \; x = \bar{x}/R. \tag{1.6}$$

The problem is then restated as

$$\frac{1}{r}\frac{\partial}{\partial r}\left(\frac{r}{T}\frac{\partial T}{\partial r}\right)+\frac{\partial}{\partial x}\left(\frac{1}{T}\frac{\partial T}{\partial x}\right)-c\frac{\partial T}{\partial x}=0, \tag{1.7}$$

$$T=1 \quad \text{at} \quad x=x_0(r), \tag{1.8}$$

$$T\to T_0 \quad \text{if} \quad x\to\infty, \tag{1.9}$$

$$\frac{\partial T}{\partial r}=0 \quad \text{at} \quad r=0, \tag{1.10}$$

$$\frac{\partial T}{\partial r}=-\epsilon\, T(T^4-T_0^4) \text{ at } r=1 \tag{1.11}$$

and is seen to feature a number of dimensionless parameters or functions

$$\epsilon=\frac{\sigma\sigma_r\bar{T}_m{}^3R}{k_m};\; c=\frac{\rho c_p vR}{k_m};\; T_0=\frac{\bar{T}_0}{\bar{T}_m};\; x_0=\frac{\bar{x}_0}{R}. \tag{1.12}$$

For a typical situation considered in [3] these parameters have the values $\epsilon=0.0967$ $c=0.01-0.1$, $T_0=3/17$, $|x_0|<0.2$, so that they can all be considered to be small. It is especially the smallness of ϵ that will enable us to deal effectively with the non-linearity of the problem. The influence of the other parameters does not seem to be very profound. Interpreting the meaning of the parameter ϵ, we see that it is a ratio of external and internal heat transfer, which means that it is a kind of Nusselt number. If it is small, the heat resistance of the crystal material is obviously relatively low.

The purpose of this paper is to investigate a method by which analytical solutions can be found for systems of the type (1.7) – (1.11), when ϵ is small. This method will be applied to the crystal growth problem as it has been described above, but the solution will be published elsewhere [3] together with experimental data in support of it. However, since the method may find application in other fields, e.g. the cooling of fins [11], it seems justified to present details of it here. We shall first study the solution of a very simple linear model problem. This will show that problems of this type may be solved by means of matched asymptotic expansions [7]. By analogy we shall apply this technique to find the solution of a related non-linear problem.

2. A model problem

As a model problem we choose Laplace's equation

$$\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}=0 \tag{2.1}$$

subject to the boundary conditions

$$x = 0: \quad T = 1; \qquad x \to \infty: \quad T \to 0,$$

$$y = 0: \quad \frac{\partial T}{\partial y} = 0; \qquad y = 1: \quad \frac{\partial T}{\partial y} = -\epsilon T. \tag{2.2}$$

It is a simple problem that bears a close resemblance to the original one. The solution can be found in a variety of ways, e.g. by separation of variables. We find

$$T = \sum_{m=0}^{\infty} \frac{4 \sin c_m}{2c_m + \sin 2c_m} e^{-c_m x} \cos c_m y, \tag{2.3}$$

where the constants c_m are the positive roots of

$$c \tan c = \epsilon. \tag{2.4}$$

Since $\epsilon << 1$, the roots may be expanded in the following manner

$$c_0 = \epsilon^{\frac{1}{2}} - 1/6\epsilon^{\frac{3}{2}} + O(\epsilon^{\frac{5}{2}}), \tag{2.5}$$

$$c_m = m\pi + \frac{\epsilon}{m\pi} + O(\epsilon^2), \quad m = 1,2,3,... \tag{2.6}$$

These expansions show that, as far as the x-dependence of the solution is concerned, there is a part that varies slowly according to the variable

$$\xi = x\epsilon^{\frac{1}{2}}. \tag{2.7}$$

Indeed, the first term of the expansion (2.3) may be written

$$\hat{T}(\xi, y) = \left\{1 + \frac{\epsilon}{6} + O(\epsilon^2)\right\} \cos\left\{y\epsilon^{\frac{1}{2}}\left(1 - \frac{\epsilon}{6} + O(\epsilon^2)\right)\right\} e^{-\xi\left\{1 - \frac{\epsilon}{6} + O(\epsilon^2)\right\}}$$

$$\sim e^{-\xi} + \epsilon e^{-\xi}\left(\frac{1}{6} + \frac{1}{6}\xi - \frac{1}{2}y^2\right) + O(\epsilon^2). \tag{2.8}$$

Strictly speaking, the expansion (2.8) is valid on a finite ξ-interval only. To extend the validity to an infinite interval we must define a strained coordinate $\xi = xc_0$ that involves the higher-order terms of the expression (2.5) for c_0. This refined approach will not be pursued here.
The remaining part of (2.3) can be expanded for $\epsilon << 1$ as well. The result is

$$\tilde{T}(x, y) = \epsilon \frac{2}{\pi^2} \sum_{m=1}^{\infty} \frac{(-1)^m}{m^2} e^{-m\pi x} \cos m\pi y + O(\epsilon^2). \tag{2.9}$$

Using the terminology of ref. [7], we may call (2.8) the outer solution of the problem defined by (2.1) and (2.2), i.e. $\hat{T} = T_{\text{outer}}$. Within the same framework the complete solution may be written

$$T = T_{\text{outer}} + T_{\text{inner}} - CP, \tag{2.10}$$

where CP is the part common to the two expansions. In the present problem this common part can be obtained by writing T_{outer}, i.e. Eq. (2.8), in the inner variable and expanding for $\epsilon \ll 1$. This leads to:

$$CP = 1 - \epsilon^{\frac{1}{2}} x + \epsilon \left(\frac{1}{6} + \frac{1}{2}x^2 - \frac{1}{2}y^2 \right) + O\ (\epsilon^{\frac{3}{2}}). \tag{2.11}$$

Since the complete solution is also equal to $\hat{T} + \widetilde{T}$, the inner expansion obviously is

$$T_{\text{inner}} = 1 - \epsilon^{\frac{1}{2}} x + \epsilon \left\{ \frac{1}{6} + \frac{1}{2}x^2 - \frac{1}{2}y^2 + \frac{2}{\pi^2} \sum_{m=1}^{\infty} \frac{(-1)^m}{m^2}\, e^{-m\pi x} \cos m\pi y \right\}$$

$$+ O(\epsilon^{\frac{3}{2}}). \tag{2.12}$$

It is well known that the expansions (2.8) and (2.12) can be obtained directly from the differential equation by application of matched asymptotic expansions [7]. This means that we do not depend on the availability of an analytical solution such as (2.3). To derive the outer expansion we merely have to recast the problem in terms of the outer variable ξ which replaces x. The terms of the expansions are matched. We shall not carry out this process for the simple model problem. However, we are led to apply this technique in the next section to solve a related non-linear problem that does not admit of an explicit analytical solution.

3. The cylindrical rod

We consider the heat flow in a semi-infinite rod with a circular cross-section which is cooled by radiation. The leading edge of the cylinder is kept at a uniform temperature. Since the singular behaviour of the problem is caused by the radiation boundary condition and since our aim is to illustrate the applicability of the method of matched asymptotic expansions, we shall carry out the analysis for a uniform thermal conductivity. Upon introduction of dimensionless variables the problem is governed by the differential equation

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial T}{\partial r} \right) + \frac{\partial^2 T}{\partial x^2} = 0 \tag{3.1}$$

and the boundary conditions are

$$\begin{aligned} &x = 0: \quad T = 1; \qquad x \to \infty: \quad T \to 0, \\ &r = 0: \quad \frac{\partial T}{\partial r} = 0; \qquad r = 1: \quad \frac{\partial T}{\partial r} = -\epsilon T^4 \quad (\epsilon \ll 1). \end{aligned} \tag{3.2}$$

Using the variable ξ defined by (2.7) and introducing the outer temperature

$$\hat{T}(\xi,r) = T(x,r),$$

we can reformulate the problem as

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \hat{T}}{\partial r}\right)+\epsilon\frac{\partial^2 \hat{T}}{\partial \xi^2}=0. \tag{3.3}$$

The boundary conditions are given by (3.2), where x should be replaced by ξ. The following asymptotic expansion is used as a possible solution:

$$\hat{T}=\hat{T}_0(\xi,r)+\epsilon\,\hat{T}_1(\xi,r)+\epsilon^2\,\hat{T}_2(\xi,r)+\ldots\,. \tag{3.4}$$

Substitution of (3.4) in the differential equation (3.3) yields

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \hat{T}_0}{\partial r}\right)=0;\ \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \hat{T}_i}{\partial r}\right)=-\frac{\partial^2 \hat{T}_{i-1}}{\partial \xi^2}\quad (i=1,2,3,\ldots). \tag{3.5}$$

Expanding the boundary condition at $r=1$ we obtain

$$\frac{\partial \hat{T}_0}{\partial r}=0;\ \frac{\partial \hat{T}_1}{\partial r}=-\,T_0{}^4;\ \frac{\partial \hat{T}_2}{\partial r}=-4\,\hat{T}_0{}^3\,\hat{T}_1 \text{ etc. for } r=1. \tag{3.6}$$

The remaining boundary conditions give obvious conditions for the perturbation functions $\hat{T}_i$.

It follows immediately that $\hat{T}_0$ is independent of r, i.e. $\hat{T}_0=A_0(\xi)$. For $\hat{T}_1$ we then find from (3.5) the general solution

$$\hat{T}_1=A_1(\xi)-\tfrac{1}{4}A_0''(\xi)\,r^2, \tag{3.7}$$

which, upon substitution in (3.6), yields a differential equation for A_0

$$A_0''=2\,A_0^4. \tag{3.8}$$

A prime stands for differentiation with respect to the argument. The solution to (3.8) is subject to the boundary conditions $A_0(0)=1$ and $A_0(\infty)=0$ so that the solution is

$$A_0=(1+a\xi)^{-\frac{2}{3}};\qquad a=\frac{3}{5^{\frac{1}{2}}}\,. \tag{3.9}$$

Up to this stage in the expansion we have been able to use the boundary condition at $\xi=0$. The next perturbation, $\hat{T}_1$, however fails to satisfy this condition. Indeed, from (3.7) we find that this boundary condition not only requires $A_1(0)$ to be zero but also $A_0''(0)=0$. From (3.9) we see that this last condition cannot be met. Therefore, Eq. (3.4) is an outer expansion that is valid in a region away from $\xi = 0$. Since at this stage we do not know what conditions to apply at $\xi = 0$, the outer expansion will contain a set of undetermined constants. It is only after we have set up an inner expansion valid for values of x that are of order unity, i.e. for $\xi \ll 1$, that we may employ a matching procedure to assign definite values to these constants.

The next few terms in the outer expansion can be obtained without undue difficulty. The general solution for $\hat{T}_2$ is seen to be

$$\hat{T}_2 = A_2(\xi) - \frac{A_1''(\xi)}{4} r^2 + \frac{1}{64} A_0^{\text{iv}}(\xi) r^4. \tag{3.10}$$

Application of the boundary condition (3.6) then yields a differential equation for A_1:

$$A_1'' - 8(1 + a\xi)^{-2} A_1 = \frac{2}{5}(1 + a\xi)^{-\frac{14}{3}}. \tag{3.11}$$

Demanding that A_1 should tend to zero if $\xi \to \infty$ we obtain the solution

$$A_1 = c_1(1 + a\xi)^{-\frac{5}{3}} + \frac{1}{24}(1 + a\xi)^{-\frac{8}{3}}, \tag{3.12}$$

where c_1 is the first of the undetermined constants.

By now it has become clear how this procedure should be continued, and we shall conclude this part of the analysis by presenting the expression for A_2, which features another undetermined constant

$$A_2 = c_2(1 + a\xi)^{-\frac{5}{3}} + \frac{5}{4} c_1{}^2 (1 + a\xi)^{-\frac{8}{3}} + \frac{1}{6} c_1 (1 + a\xi)^{-\frac{11}{3}} + \frac{119}{4320}(1 + a\xi)^{-\frac{14}{3}}. \tag{3.13}$$

We now have to tackle the problem that the outer expansion (3.4) does not satisfy the boundary condition at $\xi = 0$. In order to get some idea of how to proceed, we shall substitute $\xi = x\epsilon^{\frac{1}{2}}$ in the outer expansion and then expand for small values of ϵ. The result is

$$\begin{aligned}\hat{T} \sim\ & 1 - \frac{2}{3} ax\epsilon^{\frac{1}{2}} + \epsilon\left(x^2 - \frac{r^2}{2} + c_1 + \frac{1}{24}\right) \\ & + \epsilon^{\frac{3}{2}}\left(-\frac{8}{9}x^2 + \frac{4}{3}r^2 - \frac{5}{3}c_1 - \frac{1}{9}\right) ax + \epsilon^2\left\{\frac{22}{15}x^4 + \frac{11}{20}r^4\right. \\ & \left. - \frac{22}{5}x^2r^2 + \left(4c_1 + \frac{11}{30}\right)x^2 - \left(2c_1 + \frac{11}{60}\right)r^2 + c_2 + \frac{5}{4}c_1{}^2 + \frac{1}{6}c_1 + \frac{119}{4320}\right\} \\ & + O(\epsilon^{\frac{5}{2}}).\end{aligned} \tag{3.14}$$

This suggests that we should introduce the inner expansion

$$\tilde{T} = \tilde{T}_0(x, r) + \epsilon^{\frac{1}{2}} \tilde{T}_1 + \epsilon \tilde{T}_2 + \epsilon^{\frac{3}{2}} \tilde{T}_3 + \dots . \tag{3.15}$$

Each term of this expansion must tend to the corresponding term of (3.14) if x tends to infinity. Indeed, Eq. (3.14) is the part common to both the inner and the outer expansions. Each of the functions $\widetilde{T}_i$ satisfies the differential equation (3.1). At $x = 0$ we have $\widetilde{T}_i = 0$ with the exception of $\widetilde{T}_0(0,r) = 1$. The boundary condition at $r = 1$ must be expanded as well, i.e. at $r = 1$ we have

$$\frac{\partial \widetilde{T}_0}{\partial r} = 0; \quad \frac{\partial \widetilde{T}_1}{\partial r} = 0; \quad \frac{\partial \widetilde{T}_2}{\partial r} = -\widetilde{T}_0^4; \quad \frac{\partial \widetilde{T}_3}{\partial r} = -4\,\widetilde{T}_0^3\,\widetilde{T}_1 \quad \text{etc.} \tag{3.16}$$

It is not difficult to see that the solutions for $\widetilde{T}_0$ and $\widetilde{T}_1$ simply are

$$\widetilde{T}_0 = 1 \quad \text{and} \quad \widetilde{T}_1 = -\frac{2}{3}\,\alpha x. \tag{3.17}$$

For $\widetilde{T}_2$ we have the system

$$\nabla^2 \widetilde{T}_2 = 0, \tag{3.18}$$

$$\widetilde{T}_2(0,r) = 0; \qquad \widetilde{T}_2 \to x^2 - \frac{1}{2}r^2 + c_1 + \frac{1}{24} \quad \text{if} \quad x \to \infty,$$

$$\frac{\partial \widetilde{T}_2}{\partial r}(x,0) = 0; \qquad \frac{\partial \widetilde{T}_2}{\partial r}(x,1) = -1.$$

The constant c_1 can now be determined. Integrating the Laplace equation from $r = 0$ to $r = 1$ and using the boundary conditions we obtain

$$\frac{d^2}{dx^2} \int_0^1 r\,\widetilde{T}_2\,dr = 1,$$

from which, using the fact that $\widetilde{T}_2(0,r) = 0$, we may derive

$$\int_0^1 r\,\widetilde{T}_2\,dr = \frac{1}{2}x^2 + \text{constant} \cdot x. \tag{3.19}$$

Substituting the asymptotic result for $\widetilde{T}_2$ in (3.19) we find that

$$c_1 = \frac{5}{24}. \tag{3.20}$$

It is now a simple matter to solve the system (3.18). For example, application of the Fourier transform yields the solution

$$\widetilde{T}_2 = x^2 + \frac{2}{\pi} \int_0^\infty \left(\frac{2}{y^3} - \frac{I_0(ry)}{y^2 I_1(y)} \right) \sin xy\,dy. \tag{3.21}$$

It will now be obvious how to proceed to find the next few terms in the inner expansion. We shall refrain from giving a detailed derivation, but simply state the results

$$\widetilde{T}_3 = \alpha x\left(-\frac{8}{9}x^2 + \frac{4}{3}r^2 - \frac{11}{24}\right); \tag{3.22}$$

$$\widetilde{T}_4 = \frac{22}{15}x^4 - \frac{22}{5}x^2r^2 + \frac{6}{5}x^2 +$$

$$\frac{2}{\pi}\int_0^\infty \left(\left(\frac{4I_0(y)}{y^3I_1^2(y)} + \frac{48}{5}\,\frac{1}{y^4I_1(y)}\right)I_0(ry) - \frac{176}{5}y^{-5} + \left(\frac{12}{5} - \frac{44}{5}r^2\right)y^{-3}\right)\sin xy\,dy. \tag{3.23}$$

A uniformly valid solution to the problem (3.1)–3.2), which is called the composite expansion, is obtained by adding the expansions (3.4) and (3.15) and subtracting the common part (3.14). To be able to get an accuracy of $O(\epsilon^2)$ it will be necessary to know the value of the constant c_2. This turns out to be

$$c_2 = -\frac{575}{6912}. \tag{3.24}$$

4. Concluding remarks

In problems of the present kind the main term of the composite expansion is precisely the leading term of the outer expansion. In the previous section this term was given by Eq. (3.9). Interpreting the heat-transfer problem we see that this solution is obtained by balancing axial conduction and radiation heat transfer, assuming a uniform temperature distribution in the radial direction. In the literature this term has already been derived by Billig [8] who used exactly the same physical argument when considering the cooling of a cylindrical germanium ingot. His analysis was done for a thermal conductivity k that was inversely proportional to the absolute temperature. However, while this author used a temperature-dependent k, he mistakenly kept it constant when differentiating the axial heat flux. Since the result is of interest in the field of crystal growth, we shall present the correct result here and extend the validity of Billig's work to a cylinder of finite length x_1.

Using the system (1.7)–(1.11) with $c = 0, x_0 \equiv 0$ and $T_0 = 0$, replacing $x \to \infty$ by $x = x_1$, the first term of the outer expansion will be governed by the equation

$$\frac{d}{d\xi}\left(\frac{1}{\hat{T}_0}\,\frac{d\hat{T}_0}{d\xi}\right) = 2\,\hat{T}_0^{\,4}, \tag{4.1}$$

where ξ is again given by $\xi = x\,\epsilon^{\frac{1}{2}}$. The boundary conditions are

$$\xi = 0: \quad \hat{T}_0 = 1; \quad \xi = \xi_1: \quad \frac{d\hat{T}_0}{d\xi} = 0. \tag{4.2}$$

The second condition of (4.2) follows by expanding the radiation condition at the far end of the cylinder. This system admits of an analytical solution which is

$$\hat{T} = \left[\frac{\alpha}{\cos\{2\alpha(\xi_1 - \xi)\}}\right]^{\frac{1}{2}}, \tag{4.3}$$

where α follows from the equation

$$\alpha = \cos(2\alpha\xi_1), \quad \left(0 < 2\alpha\xi_1 < \frac{\pi}{2}\right). \tag{4.4}$$

For $\xi_1 \to \infty$ the solution assumes the simple form $\hat{T}_0 = (1 + 2\xi)^{-1/2}$

Although we have restricted our attention in this paper to the non-linearity caused by radiation, it will be clear that the method of matched asymptotic expansions will prove useful in more general cases. For example, we could extend the problem to include the effect of outside convection, both free and forced. However, if this is properly done the problem will become a conjugate one, i.e. the flow of heat both inside the crystal and in the surrounding moving gas will have to be considered. The problem of outside heat transfer alone is already rather complicated, even if attention is restricted to low-heat-resistance sheets or cylinders [9, 10]. It will therefore be rather challenging to devise a method that uses sophisticated heat-transfer models both inside and outside the cooling cylinder.

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